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A measure of sustainable efficiency in networks

Alba Valverde[1,†]

[1] *Departamento de Análisis Económico: Economía Cuantitativa, Universidad Autónoma de Madrid*

Abstract. Complex systems of interactions of different areas can be modelled by complex networks where elements are represented by nodes and the interactions by links. The architecture or topology of those networks reflect (specially when considering undirected and unweightened networks) fundamental properties of the modelled systems. In this work we analyse three main characteristics of the behaviour of these systems: its operational capacity or performance, its efficiency with respect to resources and its vulnerability. Some results of the defined sustainable efficiency measure are presented.

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† **Corresponding author:** Corresponding E-mail
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1. Introduction

Many relevant properties of complex systems (such as communication or information systems) can be described in terms of network structural or topological properties. There is an extensive literature on network performance and network efficiency where direct as well as long range interactions are taken into account. One of the most widely accepted definitions of network efficiency is given by Latora and Marchiori in [1]. The L-M *global efficiency* of a network G with n nodes is defined by:

$$Ef_{L-M}(G) = \frac{1}{n(n-1)} \sum_{i \neq j \in G} \frac{1}{d_{ij}} \tag{1}$$

where d_{ij} denotes the topological-geodesic distance between nodes i and j, that is, the minimum lenght of a path joining i and j ($d_{ij} = \infty$ if there is no path joining them), so less distant connections are more valuable than more distant ones. Note that $0 \leq Ef_{L-M}(G) \leq 1$ is a normalized measure where its maximum value is reached for the complete graph K_n on n nodes. In this sense, the Latora-Marchiori global efficiency is expressed as a percentage of network performance, $\sum_{i \neq j \in G} \frac{1}{d_{ij}}$, of what ideally could be expected, $n(n-1)$, in a complete graph as ideal case.

Since for a given number of nodes, the performance -and therefore the L-M efficiency- of the network increases with the number of edges, the need of introducing a cost evaluator function related with the number of edges arises (see [2] where the concept of economic small-world is introduced). However, as it is stated in [2], 'the target principles of construction also have to take into account the fact that resources are not unlimited'.

Vulnerability measures the stability and robustness of the global performance of the network under external perturbations (random or targeted failures or attacks). When vulnerability is defined as the relative drop in the global efficiency $Ef_{L-M}(G)$, (as in [3]), a negative value of vulnerability can appear (see [4]-[5]).

These considerations motivate the model presented in this work. First we define the operational capacity of a network as a measure of its performance in terms of the distribution distances of the network. Vulnerability is then defined as the relative drop in the operational capacity after the failure of a part of the network. Considering the restriction of limmited resources, a sustainable efficiency measure is define as the ratio of the operational capacity by a resources function instead of a cost function.

2. Operational capacity, distance distribution and vulnerability

The operational capacity is a measure of the network performance based in the structure or topology of the network by the connections of the graph. For a network $G = (V, E)$ with $|V| = n$ nodes and $|E| = m$ edges, we define its *operational capacity* by:

$$OC(G) = \sum_{i \neq j \in G} \frac{1}{d_{ij}} \tag{2}$$

Thus, $0 \leq OC(G) \leq n(n-1)$, and the maximum value is obtained for the complete network K_n.

Let G be a connected network with diameter d and let i a node of G. Let x_{i_k} be the number of nodes at distance k from i. The *distance degree sequence* of node i is $\vec{x}_i = (x_{i_1}, x_{i_2}, ...x_{i_d})$, where x_{i_k} is the number of nodes $j \neq i$ in G at distance k of i. Thus $x_{i_1} = \deg(i)$ is its degree, $x_{i_k} = 0$ for $k > \varepsilon(i)$ the eccentricity of node i, and $\sum_{k=1}^{d} x_{i_k} = n - 1$.

Let D_k be the number of pairs of nodes at distance k from one another in a G. The *distance distribution* of G is given by:

$$\vec{D}(G) = (D_1, D_2, ..., D_d) = \frac{1}{2} \sum_{i=1}^{n} \vec{x}_i$$

Thus, for a connected network G with diameter d its operational capacity is determined by its distance distribution:

$$OC(G) = 2 \sum_{k=1}^{d} \frac{D_k}{k} \tag{3}$$

which allows to give the following bounds for $OC(G)$ in terms of the diameter d, the number of nodes n and the number of links m:

- $d = 1$, $G = K_n$ and $OC(G) = n(n-1)$
- $d = 2$, $OC(G) = m + \frac{n(n-1)}{2}$
- $d = n - 1$, $OC(G) = 2 \sum_{k=1}^{n-1} \frac{n-k}{k}$
- $2 < d < n - 1$,

$$OC(G) \leq m + \frac{n(n-1)}{2} - \frac{d(d+1)}{2} + 3 + 2 \sum_{k=3}^{d} \left(\frac{d+1}{k} \right)$$

$$OC(G) \geq 2m + \frac{n(n-1)}{d} - \frac{2m}{d} - 3d + 5 + \frac{2}{d} + 2 \sum_{k=2}^{d-1} \frac{d+1}{k}$$

The *vulnerability* quantifies the network's security and stability under the effects of failures or targeted attacks. We define it as the relative fall of operational capacity, and therefore is a normalized non negative measure.

- *Node vulnerability*

$$V(G,i) = \frac{OC(G) - OC(G \setminus \{i\})}{OC(G)}$$

- *Average vulnerability*

$$V(G) = \frac{1}{n}\sum_{i\in G} V(G,i)$$

- *Maximum vulnerability*

$$V_{\max}(G) = \max_{i\in G} V(G,i)$$

3. Sustainable efficiency

For any network on n nodes, its operational capacity increases with the number of edges. In the case of a complete network K_n, the ratio between its operational capacity and its number of edges is constant equal 2. However, for a star network St_n this ratio increases with n by $\frac{n}{2}+1$. Taking into account the fact that resources are not unlimited, we define a *resources function* of a network G with n nodes and m edges by:

$$R(G) = m\left(\frac{n}{2}+1\right) \tag{4}$$

The *sustainable efficiency* of a network G is defined as the ratio of its operational capacity per the amount of the resources used.

$$Ef_S(G) = \frac{OC(G)}{R(G)} = \frac{\sum_{i=1}^{n}\left(\sum_{j\neq i}\frac{1}{d_{ij}}\right)}{\frac{1}{2}m(n+2)} \tag{5}$$

Note that $0 \leq Ef_S(G) \leq 1$, where $Ef_S(G) = 0$ iff there is no edge and $Ef_S(G) = 1$ iff $G = St_n$. Thus, in this model, best sustainable efficiency is realized in star-like configurations, which agree with results obtained by other authors (see [6]-[7]). However, the star configurations have the highest vulnerability for targeted attacks, reflecting the fact that both requierements (efficiency with respect to resources and robustness) are conflicting demands.

One of the main results of the given sustainable efficiency measure is presented in the following theorem, which states that sustainable efficiency improves generally with connectivity, although the increase of the resources value.

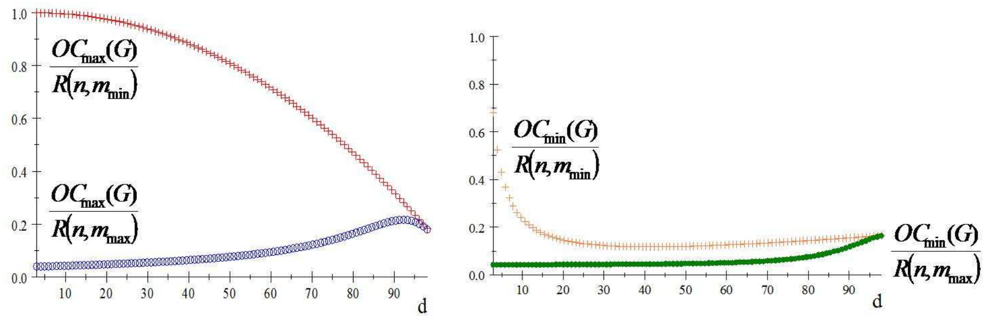

Figure 1: Ratio of the maximum operational capacity (right) and minimum operational capacity (left) by the resources function for a network G with $n = 100$ nodes and diameter $3 \leq d \leq 98$.

Theorem 3..1 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be connected graphs with n_1 and n_2 nodes respectively. Let $G = G_1 \cup G_2$ the disjoint union. Then there exists $u \in G_1$ and $v \in G_2$ such that adding an edge $\alpha = (u, v)$ in G, the resulting connected network $G' = G_1 \cup_\alpha G_2$ verifies $Ef_S(G') > Ef_S(G)$.*

References

[1] V. Latora and M. Marchiori, *Phys. Rev. Lett.* **87**, 198701 (2001).

[2] V. Latora and M. Marchiori, *New Journal of Phys.* **9**, 188 (2007).

[3] P. Crucitti, V. Latora, M. marchiori and A. Rapisarda, *Physica A* **340**, 388-394 (2004).

[4] S. Boccaletti, J. Buldú, R. Criado, J. Flores, V. Latora, J. Pello and M. Romance, *Chaos* **17**, 043110 (2007).

[5] S. Boccaletti, R. Criado, J. Pello, M. Romance and M. Vela-Pérez, *Int. J. Bifurcation Chaos* **19**, 727-735 (2009).

[6] V. Ventakasubramanian, S. Katare, P.R. Patkar and F. Mu, *Comput. Chem. Eng.* **28**, 1789-1798 (2004).

[7] M. Brede and B.J.M. de Vries, *Physics Lett. A* **373**, 3910-3914 (2009).