

Controllability of a swarm of topologically interacting autonomous agents

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Abstract. Controllability of complex networks has been the focal point of many recent studies in the field of complexity [11, 16]. These landmark advances shed a new light on the dynamics of natural and technological complex systems [11, 16]. Here, we analyze the controllability of a swarm of autonomous self-propelled agents having a topological neighborhood of interactions, applying the analytical tools developed for the study of the controllability of arbitrary complex directed networks. To this aim we thoroughly investigate the structural properties of the swarm signaling network which is the information transfer channel underpinning the dynamics of agents in the physical space. Our results show that with 6 or 7 topological neighbors, every agent not only affects, but is also affected by all other agents within the group [5]. More importantly, still with 6 or 7 topological neighbors, each agent is capable of full control over all other agents. This finding is yet another argument justifying the particular value of the number of topological neighbors observed in field observations with flocks of starlings [3].

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1. Introduction

The connectedness of the swarm signaling network (SSN), the swarm's information transfer channel, has been shown to be a sufficient condition for an agent within the swarm to affect and get affected by some if not all agents of the group [8]. However, in many occasions, one or more informed agents need to be able to drive the swarm to a certain global state, and usually

within finite time. This is better understood when considering two biological systems such as a flock of birds or a school of fish. For instance, evasive maneuvers triggered by a predator approaching or by collision avoidance collective responses are induced by one or a few agents perceiving the threat and responding to it. Those agents are said to be informed since they involuntarily have a privileged access to out-of-the-swarm informational signals. Moreover, these few agents effectively are driver agents: they are able to control the entire swarm by bringing the other agents to swiftly respond to a threat that they are not directly detecting. It is worth adding that those driver agents do not possess any “super” power of any sort but they simply temporarily become informed “leaders” as they happened to have discerned the danger first; any other agent in the swarm could be driving the group as long as it is subjected to specific external cues which are not made available globally to the whole swarm. Therefore, controllability is a vital factor for a swarm to robustly and effectively perform a dynamic collective response benefiting the majority of the group members. In this paper, we analyze the controllability of a dynamic swarm by tapping into network-theoretic concepts to represent the dynamic complex network of interactions underlying the dynamics of the collective in the case of topological interactions.

2. The Swarming Model

The model we investigate here, as a simple representation of swarming—is composed of self-propelling agents moving about a two-dimensional plane with constant speed, v_0 , similarly to the Vicsek’s model [15]. However, the neighborhood of interactions is not metric but instead is topological [3]. The topological character of the neighborhood of interactions has a tremendous impact on the properties of interagent connectivity, in particular with the induced asymmetry in the relationship whereby if agent j is in the neighborhood of agent i , then i is not necessarily in the neighborhood of j , i.e. the interaction is directed.

For simplicity, we assume that each agent i is fully characterized by one unique state variable θ_i , its velocity $\mathbf{v}_i = v_0 \cos \theta_i \hat{x} + v_0 \sin \theta_i \hat{y}$, or equivalently its velocity direction θ_i , the speed v_0 being constant. The local synchronization protocol—based on relative states that prevents any singularity such as those reported with the original Vicsek’s model [9] from occurring—is strictly equivalent to a local alignment rule, which mathematically can be stated as:

$$\dot{\theta}_i(t) = \frac{1}{|\mathcal{N}_i(t)|} \sum_{j \in \mathcal{N}_i(t)} w_{ij} (\theta_j(t) - \theta_i(t)), \quad (1)$$

where $\mathcal{N}_i(t)$ is the time-dependent set of outdegree neighbors in the agent i ’s

topological neighborhood of interaction, with cardinal number $|\mathcal{N}_i(t)|$, and w_{ij} is the binary weight of the $i-j$ communication link. Note that in some models, w_{ij} can take a more complicated form than our binary choice [4, 6, 13]. Using the k -nearest neighbor rule for the topological neighborhood of interactions, we have $|\mathcal{N}_i(t)| = k$ and the following dynamical equation for each individual agent in the swarm:

$$\dot{\theta}_i = \frac{1}{k} [(\theta_j - \theta_i) + \dots + (\theta_{j+k-1} - \theta_i)], \quad (2)$$

where $\theta_j, \dots, \theta_{j+k-1}$ are its k -nearest neighbors' velocity directions.

3. The Swarm Controllability

In order to analyze the swarm controllability, we need to identify the SSN which is the information transfer channel in the swarm underlying the dynamics of the interacting agents. The dynamics of the agents in the two-dimensional physical space is intricately coupled to the dynamics of the SSN. It is easy to verify that the SSN is a switching k -nearest neighbor digraph [1, 2, 7, 8] as agents are forced to interact with their k -nearest neighbors within the evolving swarm. Consequently, the global swarm dynamical model can be recast as

$$\dot{\Theta}(t) = \frac{1}{k} (-L)\Theta(t), \quad (3)$$

where $\Theta(t) = [\theta_1(t), \dots, \theta_N(t)]^T$ is the vector of velocity directions of all agents and L is the matrix of the graph Laplacian associated with the SSN based on the outdegree. Note that given the k -nearest neighbor rule used for the topological neighborhood of interactions, the outdegree for every single node is constant and equal to k . Figure 3.(top) depicts a snapshot of the collective migration of a swarm comprising $N = 100$ topologically-interacting agents moving in a two-dimensional square domain subjected to periodic boundary conditions. The associated signaling network is shown in Fig. 3.(middle) with the nodes representing the traveling agents at their exact location in the physical space, while the edges represent the directed topological interactions between individuals.

Recently, the field of complex networks have seen the emergence of new general theories and tools related to the controllability of such networks. The two most prominent controllability tools are: (i) the structural controllability framework developed by Liu *et al.* [11], and (ii) the exact controllability framework very recently introduced by Yuan *et al.* [16]. In applying both the exact and the structural controllability tools, one has access to the details of

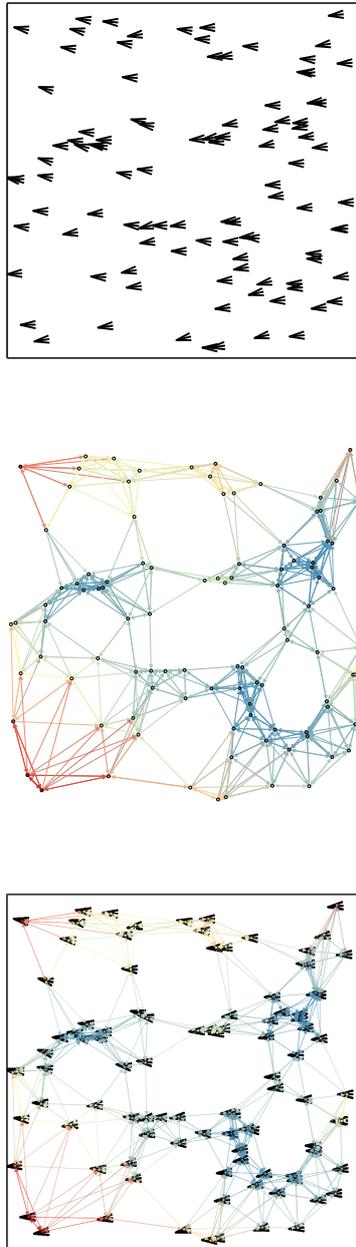


Figure 1: (Top) Physical view: snapshot of a swarm of $N = 100$ topologically-interacting individuals traveling at constant speed ($v_0 = 0.03$) in a 2D square domain (10×10) with periodic boundaries; each agent interacts topologically with $k = 7$ neighbors. (Middle) Network view: the associated swarm signaling network (SSN); the nodes and edges are colored according to the topological distance (increasing topological distance from blue to red). (Bottom) Combined view: the swarm overlaid with the SSN.

the extent of the swarm's controllability, however, it requires adapting the dynamical governing equations for the swarm to these frameworks. This is the aim of the following lemma.

Lemma. *The controllability of the system governed by Eq. (3) is equivalent to the controllability of the system*

$$\dot{\Theta}(t) = A\Theta(t), \quad (4)$$

where A is the adjacency matrix of the SSN, whose graph Laplacian matrix L appears in Eq. (3).

Proof. The number N_D of driver nodes—a.k.a. unmatched nodes—in the system governed by Eq. (3) is determined as follows:

$$N_D = \max_i \left\{ N - \text{rank} \left(\delta_i I + \frac{1}{k} L \right) \right\}, \quad (5)$$

where δ_i is the i -th eigenvalue of $\tilde{L} = -L/k$.

Given the definition of the matrix of the graph Laplacian and the topological nature of the inter-agent interactions, one can write:

$$\delta_i I + \frac{1}{k} L = \delta_i I + \frac{1}{k} (D - A) = \delta_i I + \frac{1}{k} (kI - A) = (\delta_i + 1)I - \frac{1}{k} A. \quad (6)$$

It is easy to check that both transition matrices in Eqs. (3) and (4) share the same eigenvectors. Thus, their corresponding eigenvalues are associated as

$$\delta_i = \frac{1}{k} \lambda_i - 1, \quad (7)$$

where λ_i is the i -th eigenvalue of the adjacency matrix A of the SSN. Given Eqs. (6) and (7), we can conclude that

$$\text{rank} \left(\delta_i I + \frac{1}{k} L \right) = \text{rank}(\lambda_i I - A). \quad (8)$$

□

In our swarming model, interactions among all individual agents are either “on” or “off” depending on whether the pair of agents are topologically interacting or not—or equivalently we can say that the weights of the constituent links are binary numbers, 0 or 1. This highlights the fact that link weights are not free independent parameters in our SSN model. Hence, the exact controllability framework looks suitable to be applied to our problem [16]. Figure 3. shows the results of the exact controllability analysis of the dynamical swarm at any given point in time. One can see that the number of driver

nodes decrease exponentially as k , the number of agents in the topological neighborhood, increases. We can conclude through these results that if the number of nearest neighbors reaches a value around 6 to 8—typical values for the number of topological neighbors observed by Ballerini *et al.* [3] during field experiments with bird flocks—every agent not only affects and is affected by all other agents within the group, but more importantly, is capable of full control over all other agents, i.e. the swarm.

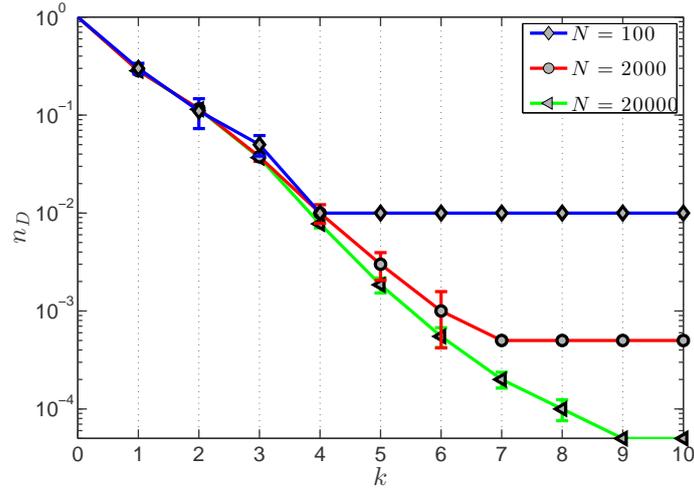


Figure 2: Density of required driver agents for a swarm with topologically-interacting members vs. the number of neighbors (k) for three different swarm populations (N). Results applying the exact controllability tool were collected for 10 distinct SSNs at each data point. The average density of driver nodes is calculated and the related standard deviations are illustrated by means of errorbars.

One concern that should be addressed regarding the above results on the number of driver nodes and the overall controllability of the swarm is associated with the dynamic nature of the SSN. Since the SSN is intrinsically a switching network—at each instant a certain number of links are broken while the exact same number of edges are created due to the motion of the agents in the physical space—one can prove that it is controllable at each instant, assuming of course a high-enough value for k , for example around 6 to 8. If that is the case, it is known from control theory associated with dynamic hybrid systems that the overall switching dynamical system is controllable [10, 14]. However, if the value of k is not large enough to have a controllable swarm at each instant, then this analysis reveals a lower bound for the control centrality of each single agent, i.e. the ability of a single agent to control the whole

swarm [12].

In either natural or artificial swarms it is more realistic to have non-binary weights for communication links in order to model the imperfection of the information transfer channel. Thus, it is necessary to consider how the swarm controllability is affected by changing the weights of edges of the SSN. Moreover, such a study would reveal the efficiency of our simple model in analyzing the swarm controllability associated with realistic cases. To that end, we further perform a structural controllability analysis of the swarm.

A system's structural controllability is to a great extent encoded in the underlying degree distribution, $p(k_{\text{in}}, k_{\text{out}})$. That is, the number of driver agents is determined mainly by the number of incoming and outgoing links each node of the SSN has, and is independent of where those links point at [11]. As mentioned before and by construction, the outdegree distribution of the SSN is a Dirac delta distribution, while its indegree distribution very much resembles the one of a k -nearest random digraph [8], namely a Poisson distribution associated with mean degree k . To allow for an analytical study of the structural controllability of the swarm, we therefore consider the following degree distributions:

$$\begin{aligned} p_{\text{out}}(k_{\text{out}}) &= \delta(k_{\text{out}} - k), \\ p_{\text{in}}(k_{\text{in}}) &= \frac{k_{\text{in}}^k}{k_{\text{in}}!} e^{-k}. \end{aligned} \quad (9)$$

Given the above discussion, the following lemma provides a key and useful result originating from the structural controllability framework [8].

Lemma. *The number of driver agents of the system governed by Eq. (4) at each time instant is given by $N_D \approx \frac{N}{2} e^{-k}$, in the large k limit.*

Figure 3. shows the required number k^* of topological agents to achieve full controllability of the swarm based on the above analytical result. In other words, Fig. 3. provides an answer to the following question: for a given swarm population N , what is the number of topological neighbors k^* required to confer to each and every single agent full controllability “powers” over all other agents. Moreover, this approximate analytical result based on the structural controllability is in very good agreement with those obtained using the exact controllability framework.

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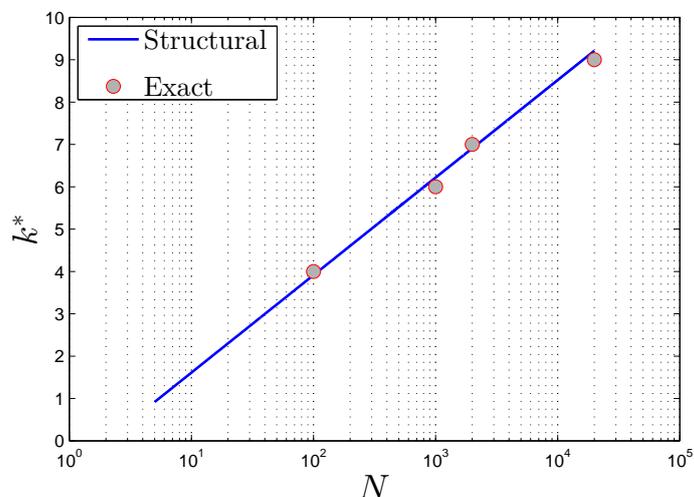


Figure 3: The required number of topological neighbors (k^*) in a swarm to reach full controllability vs. swarm size (N). The blue line corresponds to the approximate analytical result from the structural controllability analysis. The red dots refer to the result obtained with the exact controllability tool.

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