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An Algorithm for Controllability in Complex Networks

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Abstract. In this paper structural controllability of complex networks is analyzed. A new algorithm is proposed which constructs a structural control scheme for a given network by avoiding the absence of dilations and by guaranteeing the accessibility of all nodes. Such accessibility is solved via a *wiring* procedure; this procedure, based on determining the non-accessible regions of the network, has been improved in this new proposed algorithm. This way, the number of dedicated controllers is reduced with respect to the one provided by previous existing algorithms.

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1. Introduction

Complex systems composed by many interconnected elements can be studied by analyzing the underlying network (or graph) which defines these interconnections [1]. This network gathers some fundamental structural system properties which in general do not depend on the specific values of system parameters. When addressing the control of this type of complex systems, one has to take into account both node dynamics as well as their connectivity [3, 4, 16]; in this paper we focus on structural aspects which define the basic information for grounding a detailed system control analysis.

Standard control theory for dynamical systems defined by difference or differential equations [12, 19] is aimed at modifying the system internal state via the injection of appropriate system inputs [5]. As mentioned earlier a robust analysis is focused on structural aspects. In this context, structural controllability theory has been developed [9, 10, 14, 15] for characterizing the implications of system internal connectivity in its state space controllability. The main results of structural controllability theory refer to the structural properties of some matrix associated with the system as well as some properties of the system associated graph. Equivalent results have also been obtained for the dual structural observability problem [11, 17, 18]. Based on this duality, we focus on the controllability problem.

The structural controllability of a system is characterized by its internal connection structure and the set of inputs defined in such system. Given the network defining the system connectivity and a set of inputs, a system is structurally controllable if and only if all nodes (or vertices) are accessible from the inputs and the network does not have dilations [9].

The paper is organized as follows. Section 2. some fundamental results concerning bipartite graphs, perfect matchings and dilations in the context of structural controllability. This allows to address the problem of finding the minimum number of controller inputs which guarantee system structural controllability of the overall network. Precisely, graph theoretical tools are mainly employed to find such minimum number of control inputs for avoiding dilations, where the concept of Matching becomes extremely useful [10, 15]. Additionally, a *wiring* procedure is developed to implement accessibility of all nodes while preserving the minimum number of control inputs. The paper continues in Section 3. presenting a new algorithm aimed to reduce the number of wires, which implies the use of a lower quantity of dedicated inputs. Finally, concluding remarks are stated in Section 4..

In the following section some of the fundamental results concerning system structural controllability are presented.

2. Structural controllability fundamental results

In Section 1., it is stated that a network is structurally controllable if and only if all nodes are accessible from the inputs and the network has no dilation [10]. In this Section we are analyzing both conditions.

Let us begin with some definitions and an important result from Graph Theory:

Definition 1 (Graph) *A graph (or network) is defined by* $G := (V, E)$*, a pair of sets where* V *is the set of vertices (or nodes) and* E *the set of edges (or links). Every edge belonging to* E *is defined as a pair of vertices* $(v_i, v_j) \in E$ *where* $v_i, v_j \in V$*.*

Remark 1 *In a directed graph, the pair defining an edge is ordered.*

Definition 2 *(Bipartite Graph) A bipartite graph is a particular case, defined as* $G_b := (X, Y, E_b)$ *where* X *and* Y *are two (potentially) different sets, and* E_b *is the set of edges linking one element of* X *with an element of* Y*, i.e* $E_b = \{(v_i^X, v_j^Y) : \ v_i^X \in X, v_j^Y \in Y\}$*.*

Definition 3 $T(S)$ *is defined as the set of vertices that point to any vertex* $v_i \in S$*. In a bipartite graph, given a set of nodes* $S \subseteq Y$*, we define* $T(S) \subseteq X$ *such that* $T(S) = \{v_i^X \in X : \ \exists (v_i^X, v_j^Y) \text{ with } v_j^Y \in S\}$*.*

Theorem 1 (Hall) *A bipartite graph* $G_b := (X, Y, E_b)$ *has a matching which covers every vertex in* X *if and only if*

$$|T(S)| \geq |S| \ , \ \forall S \subseteq Y \tag{1}$$

Hall's theorem provides a criterion for bipartite graphs to have a perfect matching, as long as they also verify:

$$|X| = |Y| \tag{2}$$

This condition on the existence of a perfect matching is condensed in the following corollary:

Corollary 1.1 *A bipartite graph* $G_b := (X, Y, E_b)$ *has a perfect matching if and only if it verifies both* (1) *and* (2)*.*

With these definitions and results, we can face the dilation problem.

2.. 1 Dilation and Matching

By definition, a network presents a dilation if we can find a subset $S \subset V$ of the vertices of the graph that verifies:

$$|S| > |T(S)| \tag{3}$$

Given a graph $G := (V, E)$ one can define its associated bipartite graph $G_b := (V^+, V^-, E_b)$ where $V^+ = V^- = V$ and E_b is constructed as follows

$$E_b = \{(v_i^+, v_j^-) : \exists (v_i, v_j) \in E\}$$

Defined this way, $G_b : (V^+, V^-, E_b)$ will obviously verify (2). Therefore, a network has no dilation if and only if it has a perfect matching.

Alternatively, based on the pioneering work of [9], the Maximum Matching (MM) Algorithm is proposed in [10] as a good tool to determine the minimum number of inputs required to guarantee that there are no dilations in a network. Precisely, the number of required inputs would correspond with the number of vertices not matched by an edge in the maximum matching of the network, if we just consider the dilation condition.

2.. 2 Accessibility and wiring

However, as stated earlier, the matching criterion verifies the condition of no dilation in the network, but not the accessibility condition.

Every graph $G(V, E)$ has an associated *condensation* description which corresponds to a directed acyclic graph whose nodes represent the Strongly Connected Components (SCCs) of G and whose edges represent the existence of at least one edge in the original graph connecting the corresponding SCCs. The nodes with zero in-degree in the condensation, will correspond to those SCCs which cannot be accessed from the rest of the network, the so-called *non-top linked* SCCs.

In order to guarantee the accessibility of the whole network from the controllers, there must be at least one directed path from a controller to every non-top linked SCC of the network. However, a maximum matching M^* might not leave any unmatched node within a non-top linked SCC (for example, think of matching containing a cycle that spans the whole SCC).

In these cases, it is necessary to add new connection edges from the calculated inputs to the non-accessible SCCs; we call this process *wiring.* Note that this process keeps the number of control inputs unchanged.

In the following Section, a version of the MM Algorithm is proposed and analyzed. In addition, a new wiring algorithm is developed, in order to find the structures matched by the MM that are not accessible from the inputs, and to create the required external edges from the inputs to a node of such structures.

3. The combined MM and wiring algorithm

3.. 1 The Hopcroft-Karp algorithm

The study of complex systems is focused on emergent properties. Particularly in complex networks, these only take place when networks are formed by a large amount of nodes and edges. Under these circumstances, the time efficiency of the Maximum Matching (MM) becomes critical, because not only algorithms taking super-polynomial time cannot be applied, but also running algorithms taking polynomial time becomes challenging when complexity goes above cubic time. In this paper, the Hopcroft-Karp algorithm has been selected for finding MMs, which runs $O(\sqrt{V}E)$ time [7].

While Hopcroft-Karp algorithm directly deals with non-directed bipartite graphs, it can also be applied to general non-bipartite directed graphs, provided an appropriate transformation is previously performed. As mentioned earlier, from any directed graph $G := (V, E)$ it is possible to define its associated bipartite graph $G_b := (V^+, V^-, E_b)$ where $V^+ = V^- = V$. Note that the edges in the bipartite graph adjacent to any $v_i^+ \in V^+$ represent the out-links of $v_i \in V$ and the edges adjacent to $v_i^- \in V^-$ represent the in-links of v_i.

Let M_b be a matching of G_b. If $(v_i^+, v_j^-) \in M_b$, then $(v_i, v_j) \in M$ where M is the corresponding matching of G. $(v_i^+, v_j^-) \in M_b$ makes nodes v_i^+, v_j^- to be matched on the non-directed bipartite graph G_b while $(v_i, v_j) \in M$ makes v_j to be matched on the directed graph G. Therefore, once we compute the Hopcroft-Karp algorithm over G_b and obtain a maximum matching M_b^* (where $*$ stands for maximum), the unmatched nodes of the set V^- will be the unmatched nodes of the matching M^* in G.

According to the minimum inputs theorem [10], the nodes unmatched by a maximum matching form a valid set of driver nodes of the network, so a controller node has to be linked to each one of these driver nodes.

3.. 2 The accessibility and wiring algorithms

As stated in section 2.. 2 it is not enough to obtain the maximum matching M^* of the graph G to find the nodes that need to be controlled. Placing the controllers on the unmatched nodes only guarantees the absence of dilations on the graph, making it necessary to additionally check for the existence of inaccessible nodes from the controllers of the network. From section 2.. 2 it follows that these inaccessible nodes are located in those non-top linked SCCs that do not contain any unmatched node. The method proposed here consists on adding additional wires from the control inputs to such SCCs.

The determination of the SCCs can be done by applying the Tarjan algorithm [8], which runs in $O(|V| + |E|)$ time. Once the SCCs have been found, it is necessary to obtain those that are non-top linked (the roots of the condensation), selecting those SCCs whose nodes do not receive any link from outside the SCC. The worst case scenario implies checking every in-link of the nodes of each SCC, meaning that at most $|E|$ steps will be necessary, thus the Tarjan complexity dominates the finding of the non-top SCCs.

A maximum matching, M^*, is then obtained via the Hopcroft-Karp algorithm, as explained on section 3.. 1, which runs in $(\sqrt{|V|}|E|)$ time. Once both M^* and the non-top SCCs have been found, additional wires are added to those non-top linked SCCs which lack of any unmatched node. Putting all the steps together, the search of the nodes that need to be controlled on a graph G can be summarized in the following steps:

1. Find the non-top linked SCCs of the network.
2. Find the maximum matching M^* of G and link a new different controller node to each unmatched node.
3. Find those non-top linked SCCs that do not contain a controller node and add a wire from any controller to each of them.

4. Concluding remarks

An algorithm for analyzing structural controllability of complex networks has been presented. The algorithm combines a Maximum Matching search and a new *wiring* algorithm to efficiently determine the number of required controllers as well as the new required connections in the network. Further work is being developed headed to minimize the number of these additional required wires.

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