IJCSS International Journal of Complex Systems in Science

Int. J. Complex Systems in Science
vol. **3(1)** (2013), pp. 107–112

Observing Dynamical Processes in Multiplex Networks by Using Edge Correlation

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Abstract. The recent emergence of the new representation of interconnected layers of networks, called multiplex networks, refocused the research in complex networks. The analysis of these networks requires redefinition of the dynamical processes and topological properties present in one-layered graphs. The main focus in this work is the relation between consensus and synchronization defined on multiplex graphs and edge correlation of the graph layers. We show numerically that the information of edge correlation between layers offers substantial insights in the complex multiplex structure, thus, contributing to simplified and faster analysis and design of multiplex graphs with desired consensus or synchronization properties.

Keywords: multiplex networks, consensus, synchronization, edge overlap, correlation
Received: December 5th, 2013
Published: December 31th, 2013

1. Introduction

Recently, network science has shifted from analysing one-layered networks to multiplex networks, i.e. composite graph where each node contributes to the dynamical processes in several layers of the graph. The analysis of multiplex graphs is strived towards giving structural and functional representation of the interconnected networks that influence each other. Multiplex graphs provide a mathematical rationale for the analysis of many social networks (e.g. different types of social ties in Facebook, Twitter), transportation networks (e.g.

locations are layered by the type of the transport used), biological networks (e.g. parallel signalling channels in biochemical networks) and many more [1].

Since the novel structure of interconnected dynamical network changes the way nodes react to "local" information, it is exceedingly important to assess the behaviour of processes like information spreading and synchronization on multiplex graphs. The process where all nodes share local information in order to asymptotically reach a certain common state is called consensus [2]. Using local distributed protocol, nodes' information states converge to an agreement value, which is a function of the initial states (e.g. average, weighted average, maximum, minimum, etc.). Similar concepts include state agreement, rendezvous, synchronization and gossip algorithms in computer science [3].

In order to proffer deeper insight in the novel characteristics of multiplex graphs and their effect on the dynamics of network processes, we scrutinize the connection between the edge correlation and the convergence rate (i.e. the speed) of reaching consensus and synchronizability in multiplex networks. Our conjecture is that this connection will give easy and fast observation of the expected dynamical behaviour of the multiplex network and can contribute to the design of a multiplex network with desired consensus/synchronization properties.

2. Multiplex Networks

A multiplex network is defined as graph with MN nodes and M layers. Let us assume only two layers where $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are simple and undirected graphs such that $V_1 = \{v_1, \dots, v_n\}$ and $V_2 = \{u_1, \dots, u_n\}$. A multiplex graph $G(V, E)$ is defined by $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup \{v_1, u_1\} \cup \dots \{v_n, u_n\}$. We can define the information states $x = [x_1, \dots, x_n]^T$ and $y = [y_1, \dots, y_n]^T$ as n-dimensional vectors where x_i is assigned to a node i in V_1 and y_i is assigned to the corresponding node i in V_2. Let A_1 and A_2 be the adjacency matrices associated with graphs G_1 and G_2, respectively, where $a^1_{ij} = 1$ if $a^1_{ij} \in E_1$ and 0, otherwise, and $a^2_{ij} = 1$ if $a^2_{ij} \in E_2$ and 0, otherwise. We define a multilayer adjacency matrix A^m with elements $a^m_{ij} = 1$ if $a^1_{ij} = 1$ or $a^2_{ij} = 1$ and $a^m_{ij} = 0$ if no edge connects nodes i and j in any of the layers. The multilayer degree of node i can be defined as $k^m_i = \sum_j a^m_{ij}$, or the total number of distinct edges that i has in any of the layers. As shown in Fig. 1 the multiplex network is formed using coupling links that interconnect corresponding nodes between layers. However, the couplings between layers are only between a node and its counterpart(s) in the other layer(s). Here, we assume that these coupling links (shown with dotted lines) have constant weight c. The coupling weight defines how strong is the coupling influence between the layers. The Laplacian matrix L_G, of the multiplex graph, or also

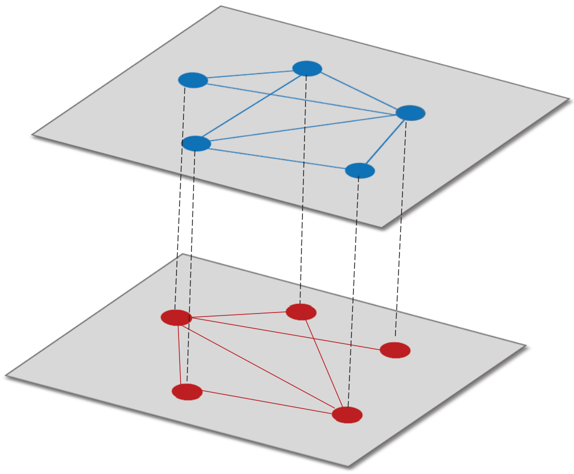


Figure 1: A multiplex network example

called supra-Laplacian matrix, with coupling weight c can be defined as in [1]:

$$L_G = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} + c \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \quad (1)$$

where L_1 and L_2 are the corresponding Laplacian matrices for G_1 and G_2, defined as $L = D - A$, where D is diagonal matrix with elements $d_{ii} = \sum_j a_{ij}$ and I is the identity matrix. The Laplacian matrix of an undirected connected graph is row stochastic with eigenvalues, $0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots \leq \lambda_N$.

A well known property of the eigenvalues of the Laplacian matrix is their effect on the convergence and stability properties of the consensus and synchronization. The second smallest eigenvalue λ_2, characterizes the convergence speed of the consensus algorithm, and maximizing this value leads to a faster consensus [2]. In synchronization, the local stability of the synchronization manifold of phase oscillators is analysed using the ratio $\rho = \lambda_N/\lambda_2$, the smaller the ratio ρ, the easier the synchronization [4, 5]. For thorough analysis of the eigenvalues of the supra-Laplacian matrix please refer to [1, 6].

3. Edge Correlation

The overlap of the edges in different graphs with same nodes has been used to define a measure for graph correlation. In [7], authors define local and global overlap between graphs to describe multiplex ensembles of correlated and uncorrelated graphs. The entropy of the multiplex, and similar participation coefficient using edge overlap and node degree are used in [8] to assess the role of the node in a multiplex network.

In order to gauge the edge correlation in multiplex networks, we define the global participation coefficient which quantifies the participation of nodes in M layered multiplex network as:

$$P = \frac{1}{N} \sum_{i=1}^{N} \sum_{\alpha=1}^{M} \frac{k_i^{[\alpha]}}{k_i^{[m]}} - 1, \tag{2}$$

where $k_i^{[\alpha]}$ is the degree of node i in layer α, $k_i^{[m]}$ is the multilayer degree of node i, defined previously, and N is the number of nodes in each layer. Completely uncorrelated layers have $P = 0$, which means that no edge in one layer is present in the others, and 1 means that all the edges in all the layers are same, i.e. the layers are completely correlated.

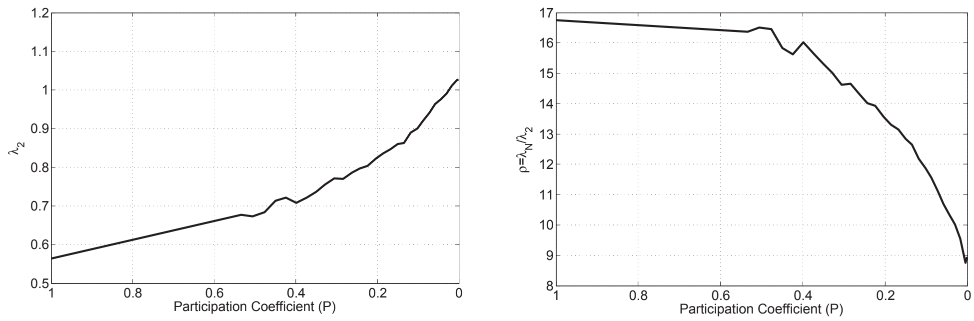
4. Numerical Results

In this Section, we investigate the connection between the defined participation coefficient and convergence rate of the consensus algorithms by measuring the second smallest non-zero eigenvalue (λ_2, the higher the value, the faster the convergence rate) of the supra-Laplacian matrix [1]. Similarly, for the synchronization process, we calculate the ratio of $\rho = \lambda_N/\lambda_2$ (smaller values resulting in faster synchronization).

Repeated simulations were performed on multiplex networks with two layers. The layered graphs were based on random regular topology and had equal number of links. The coupling weight c was set to 1, since in the region $c \approx 1$, as N tends to infinity, the ratio ρ is independent of c and has the smallest value [1, 6].

In order to assess the dependence between the participation coefficient and the dynamical processes on multiplex network we use a rewiring algorithm in one of the layers. The algorithm controls the participation coefficient and measures the required values for λ_2 and ρ. The objective function of the rewiring algorithm is to minimize the participation coefficient (i.e. minimize the edge correlation) between the layers. In addition, the rewiring algorithm retains the same number of edges in each layer and it keeps the network connected.

The presented results in Fig.2 show that by decreasing the overlap between the layers (i.e. by decreasing P) the resulting multiplex network has higher values for λ_2 (Fig. 2a) and smaller values for the ratio ρ (Fig. 2b). For the initial multiplex network, consisted of 2 identical completely overlapped random regular networks (i.e. with participation coefficient $P = 1$), λ_2 is around 0.58 and ρ is around 16.8. After the rewiring, the multiplex network is consisted of 2 random regular networks which are completely non-overlapped (i.e. multiplex network with $P = 0$). This network is characterized by $\lambda_2 \approx 1.03$ (relative increase of 78% compared to the initial network) and $\rho \approx 8.9$ (relative decrease of 53% compared to the initial network).



(a) (b)

Figure 2: The influence of the participation coefficient (P) on (a) the second smallest eigenvalue λ_2 and on (b) the eigenratio $\rho = \lambda_N/\lambda_2$

Our extensive simulations show numerically that the convergence rate for the consensus (represented by λ_2) and synchronizability (represented by ρ) increases as the participation coefficient between the layers decreases.

5. Conclusions

Our precursory results show that the value of the global participation coefficient can give insights to the convergence speed of consensus and/or synchronizability of the multiplex network, as a first-order approximation compared to the more complex calculations of the network eigenvalues.

Moreover, we have shown that by using simple rewiring (perturbation) in one of the layers in a multiplex network, with an objective to minimize the edge correlation, the convergence speed of the consensus algorithm or the synchronizability of the network can be significantly increased.

Acknowledgements

The work in this paper was partially financed by the EUROWEB Project (http://www.mrtc.mdh.se/euroweb/) funded by the Erasmus Mundus Action II programme of the European Commission and the Faculty of Computer

Science and Engineering, at the University "Ss. Cyril and Methodius" in Skopje.

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