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Convergence of Weighted-average consensus for undirected graphs

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Abstract. In this note we address the problem of reaching a consensus in an undirected network where the nodes interchange information with their neighbors. Each node is provided with a value x_i^0 and a weight w_i. The specific goal of the consensus is that each node will be aware of the weighted-average consensus value, $\frac{\sum_i w_i x_i^0}{\sum_i w_i}$, in a distributed way, that is to say without a central control. We show the applicability of a theoretical result about reaching a consensus following an iterative algorithm.

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1. Introduction

Let $G = (V, E)$ be a graph, with $V = \{v_1, v_2, \dots, v_n\}$ a non-empty set of n vertices (or nodes) and E a set of m edges. Each edge is defined by the pair (v_i, v_j), where $v_i, v_j \in V$. The adjacency matrix of the graph G is $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ such that $a_{ij} = 1$ if there is an edge connecting node v_i to v_j, and 0, otherwise. The degree d_i of a node i is the number of its links, i.e., $d_i = \sum_{j=1}^{n} a_{ij}$. We define the Laplacian matrix of the graph as $L = D - A$ where D is the diagonal matrix with the degrees. $D = diag(d_1, d_2, \dots, d_n)$.

2. Weighted-average consensus

Let G be an undirected graph. Let $\mathbf{x}^0$ be a (column) vector with the initial state of each node. Let $\mathbf{w} = [w_1, w_2, \dots, w_n]^T$ a vector with the weight associated to each node. The following algorithm (see [1]) can be used to obtain the value of the weighted-average consensus (that is, a common value for all the nodes, reached by consensus)

$$W\dot{x} = -Lx \tag{1}$$

with $W = diag(w_1, w_2, \dots, w_n)$. A discretized version of (1) is

$$x_i^{k+1} = x_i^k + \frac{\epsilon}{w_i} \sum_{j \in N_i} a_{ij}(x_j^k - x_i^k), \quad \forall i \in N, \quad k = 0, 1, 2, \dots \tag{2}$$

where $\epsilon > 0$, and N_i denotes the set of neighbors of node i. The matrix form of (2) is

$$\mathbf{x}^{k+1} = P_w \mathbf{x}^k \quad k = 0, 1, 2, \dots \tag{3}$$

where $P_w = I - \epsilon W^{-1} L$. From (2) it follows that

$$\mathbf{x}^k = P_w^k \mathbf{x}^0, \quad k = 1, 2, \dots \tag{4}$$

3. Convergence of Weighted-average consensus for undirected graphs

In [2] we prove the following: Let G be a connected undirected graph. If $\epsilon < min_{i \in N}(w_i/d_i)$ then the scheme (5) converges to the weighted-average consensus given by $\mathbf{x}_w = \alpha\, \mathbf{e}$, with $\mathbf{e}$ the all-ones vector and

$$\alpha = \frac{\sum_i w_i x_i^0}{\sum_i w_i}. \tag{5}$$

This result is useful for obtaining a weighted-average consensus in a distributed way, as we show in the next example.

4. Examples

To show the applicability of the previous result we run three series experiments. Denoting $\epsilon_o = min_{i \in N}(w_i/d_i)$, the experiments are characterized by the following:
Type I: Verify the condition $\epsilon < \epsilon_0$ and therefore converge
Type II: Do not verify the condition $\epsilon < \epsilon_0$ but they converge
Type III: Do not verify the condition $\epsilon < \epsilon_0$ and do not converge

In all the experiments the initial values x_i^0 are taking randomly distributed among the nodes, with values $x_i^0 \in [0,1]$. The weights are randomly distributed with $w_i \in (0,1)$. In Figure 1 we show examples of type II and type III for a network of $n = 10^3$ nodes constructed with preferential attachment.

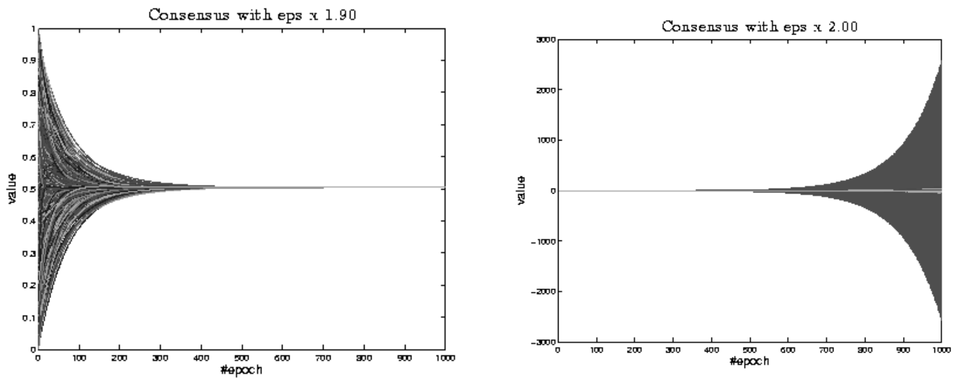
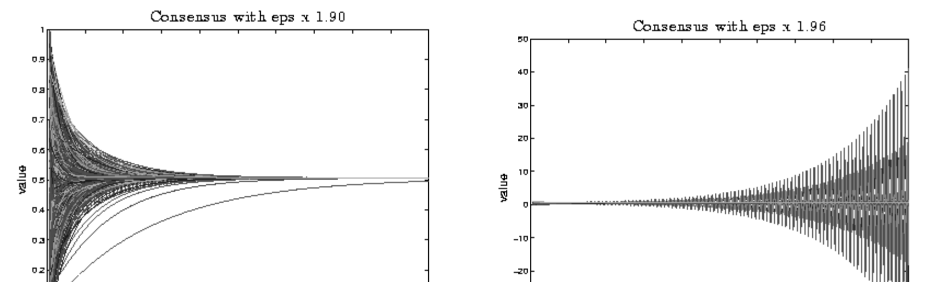


Figure 1: Example of experiments of Type II (left, $\epsilon = 1.9\epsilon_o$) and Type III ($\epsilon = 2.0\epsilon_o$) using a network generated with preferential attachment. Horizontal axis shows number of iterations, while vertical axis shows the values of x_i^k for each node $i = 1, \dots n$.

In Figure 2 we show examples of type II and type III for a network of $n = 10^3$ nodes with links randomly generated.



Examples of experiments of Type II (left, $\epsilon = 1.90\epsilon_o$) and Type III ($\epsilon = 1.96\epsilon_o$) using a random network. Horizontal axis shows number of iterations, while vertical axis shows the values of x_i^k for each node $i = 1, \dots n$.

5. Conclusions

We have shown examples of the applicability of a theoretical result about the converge of weighted-average consensus. The results shown that there is a zone (of values of ϵ) beyond the convergence limit that shows convergence. This is in accordance with the theoretical result, since we are illustrating a sufficient criterion. These results show that future work might be done to refine the convergence criterion.

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