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Stratification of the space of matrices defining linear dynamical systems

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Abstract. Given the space of matrices defining a linear system we introduce a stratification in that space attending to whether they have the same Drazin inverses.

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1. Introduction

Given a dynamical system, the effect of small perturbations in the entries of these matrices has been studied by different authors (see, for example, [1], [5],[7], [8], [9]). Since there are some specific differences of the behaviour of the nilpotent and non-nilpotent parts, we propose here a new stratitication, taking this into account.

A finite partition in the space of square matrices is obtained. Matrices belonging to the same strata are those having identical Segre characteristic of the non-nilpotent part in Jordan reduced form. This stratification is different than the one proposed by V.I. Arnold (see [1]) and studied by C.G. Gibson (see [9]), and no relationship between strata holds in general.

Further study will be devoted to the analysis of perturbations of the matrices defining the systems.

The structure of the paper is as follows.

In Section 2, we give a brief resum on the main topics involved in the paper.

In Section 3, we consider a new partition in the space of square matrices and prove that it is a stratification.

In Section 4, we provide the relationship between this stratification and Drazin pseudo-inverse.

In Section 5, we apply this stratification to dynamical systems.

Finally, some conclusions are presented.

2. Motivation and preliminaries

Let us consider a linear dynamical system

$$E\dot{x}(t) = Ax(t) + Bu(t)$$

In the case where E is an invertible matrix, pre-multiplying the state equation by E^{-1} we obtain

$$\dot{x}(t) = A_1x(t) + B_1x(t)$$

If E is not invertible, and assuming that the matrix pencil $\lambda E + A$ is regular (to ensure the system has an unique solution) the system splits in a slow and a fast parts, from the Kronecker reduced form of the matrix pencil as we show below.

Let us consider the Kronecker reduced form of the matrix pencil $\lambda E + A$:

$$\lambda E^c + A^c \quad = Q(\lambda E + A)P \quad = \lambda \begin{pmatrix} I_\nu & 0 \\ 0 & \mathcal{N} \end{pmatrix} + \begin{pmatrix} G & 0 \\ 0 & I_{n-\nu} \end{pmatrix}$$

with $\mathcal{N}$ a nilpotent matrix and G a matrix in Jordan reduced form,

$$G = \begin{pmatrix} J & \\ & N \end{pmatrix}$$

where $\det(J) \neq 0$ and N is a nilpotent matrix.

If we left pre-multiply the state equation $E\dot{x}(t) = Ax(t) + Bu(t)$ by Q and apply a state variables change: $\overline{x}(t) = P^{-1}x(t)$, we obtain:

$$\lambda \begin{pmatrix} I_\nu & 0 \\ 0 & \mathcal{N} \end{pmatrix} \dot{\overline{x}}(t) = \begin{pmatrix} G & 0 \\ 0 & I_{n-\nu} \end{pmatrix} \overline{x}(t) + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u(t).$$

From the decomposition above, the slow and fast subsystems related to the equation are thus obtained:

$$\dot{x}_1(t) = Gx_1(t) + B_1u(t)$$

$$\mathcal{N}\dot{x}_2(t) = x_2(t) + B_2u(t)$$

Note that in the case of a non-singular linear system, only the slow subsystem appears.

The solution of the fast part is well-known (see [4]). Denoting by h the nilpotency index of matrix $\mathcal{N}$, the solution is:

$$x_2(t) = -\sum_{i=0}^{h-1} \mathcal{N}^i B_2 u^{(i)}(t)$$

The slow subsystem

$$\Sigma_S \qquad \dot{x}_1(t) = Gx_1(t) + B_1u(t)$$

can be divided again in two subsystems

$$\Sigma_J \qquad \dot{y}(t) = Jy(t) + B_1^1u(t)$$

$$\Sigma_N \qquad \dot{z}(t) = Nz(t) + B_1^2u(t)$$

The solutions to the subsystems above are:

$$y(t) = e^{Jt}y_0 + e^{Jt}\int_0^t e^{-J(t-\tau)}B_1^1u(\tau)d\tau$$

$$z(t) = e^{Nt}z_0 + e^{Nt}\int_0^t e^{-N(t-\tau)}B_1^2u(\tau)d\tau$$

Different methods and algorithms have been proposed to compute the exponential of a matrix, but none of them is absolutely solid for stability and efficiency. In the case of a nilpotent matrix, its exponential matrix can be computed as a polynomial. Nevertheless, when studying linear systems, many difficulties arise from nilpotent parts. This is the reason we consider separately the invertible and the nilpotent parts of matrix G.

3. Stratification in the space of square matrices

We establish first some notation.

We will denote by $M_n(\mathbb{C})$ the set of square matrices of order n with coefficients in $\mathbb{C}$ and by $Gl_n(\mathbb{C})$ the set of all invertible matrices of order n.

We will also denote by α the following action.

$$\begin{aligned} \alpha : Gl_n(\mathbb{C}) \times M_n(\mathbb{C}) &\longrightarrow M_n(\mathbb{C}) \\ (S, M) &\longrightarrow SMS^{-1} \end{aligned}$$

Given any matrix $M \in M_n(\mathbb{C})$, its orbit under this action will be denoted by $\mathcal{O}(M)$.

We will construct a stratification such that all matrices in a same stratum will have the same Segre characteristic of the invertible part in its Jordan reduced form. It will actually be a finite partition of the space of square matrices.

Notice first the following statement, which can be easily proved.

Lemma 1 $M_n(\mathbb{C}) = \bigcup_{1\leq\mu\leq n} \mathcal{O}(\mathrm{diag}(J,N))$ *where* J *and* N *are matrices in Jordan reduced form, order of matrix* J *is equal to* μ, $0 \leq \mu \leq n$, $\det(J) \neq 0$ *and* N *a nilpotent matrix.*

In [1], given the set of square matrices or order μ, the set of those having the same Segre characteristic σ, which we will denote by $E_\mu(\sigma)$ was proved to be a differentiable manifold. The properties of the stratification of $M_\mu(\mathbb{C})$ were studied in [7].

Remark 1 *It is easy to see that the set*

$$E'_\mu(\sigma) = \{M \in M_\mu(\mathbb{C}) \mid SegreChar(M) = \sigma, \det(M) \neq 0\}$$

is also a differentiable manifold, because it is an open set of the differentiable manifold $E_\mu(\sigma)$.

Let us denote by $Nil_{n-\mu}(\mathbb{C})$ the set of nilpotent matrices of order $n-\mu$, for $1 \leq \mu \leq n$.

Remark 2 *For each matrix $N \in Nil_{n-\mu}(\mathbb{C})$, the set $E'_\mu(\sigma) \times N$ is also a differentiable manifold.*

Taking this into account, we can state the following result.

Proposition 1 *The set*

$$\bigcup_{N \in Nil_{n-\mu}(\mathbb{C})} E'_\mu(\sigma) \times N$$

is a differentiable manifold.

Proof. After the remarks above, it suffices to take into account that all the manifolds $E'_\mu(\sigma) \times N$ are disjoint and have the same dimension (their dimension is equal to $\dim E'_\mu(\sigma)$).

We will denote by $\overline{S}(\mu, \sigma, N)$ the image under the action α of the set $\bigcup_{N \in Nil_{n-\mu}(\mathbb{C})} E'_\mu(\sigma) \times N$.

Using this notation,

$$M_n(\mathbb{C}) = \bigcup_{\substack{1 \leq \mu \leq n \\ N \in Nil_{n-\mu} \\ \sigma = SegreChar \text{ of a matrix of order } \mu}} \overline{S}(\mu, \sigma, N)$$

The number of sets $\overline{S}(\mu, \sigma, N)$ is finite. Thus the partition above of the space of square matrices of order n is a finite partition. If they are proved to be differentiable manifolds, we would have obtained a (finite) partition of the space of square matrices.

Theorem 1 *The sets $\overline{S}(\mu,\sigma,N)$ are differentiable complex manifolds.*

Proof. We recall that the homogeneity along the orbits under action α holds and we can restrict ourselves to the case where matrices are in Jordan canonical form. Moreover, any miniversal deformation T is a manifold tranverse to all orbits. Then the set $S(\mu,\sigma,N)$ is a differentiable manifold if, and only if, so is its intersection with a slice, for example, T, $\overline{S}(\mu,\sigma,N)\cap T$. Note that the slice T is, in turn, diffeomorphic to a cartesian product of slices at the Jordan blocks corresponding to each eigenvalue.

Let us assume that J has r different eigenvalues having the blocks corresponding to each eigenvalue orders $\mu_1,\dots,\mu_r$ and Segre characteristics $\sigma_1,\dots,\sigma_r$.

Straightforward calculations show that $S(\mu,\sigma,N)\cap T$ is diffeormorphic to the cartesian product $(E_{\mu_1}(\sigma_1)\cap T_1)\times(E_{\mu_r}(\sigma_r)\cap T_r\times(\mathcal{O}(N)\cap T_N)$, where $T_1,\dots,T_r$ are the slices at the constituent Jordan blocks of matrix J, and T_N the slice corresponding to the nilpotent block (T is diffeomorphic to the cartesian product $T_1\times\ldots\times T_r\times T_N$).

Corollary. The sets $\overline{S}(\mu,\sigma,N)$ provide a finite stratification of the space $M_n(\mathbb{C})$.

4. Relationship with Drazin inverse

Given a matrix $A\in M_{m\times n}(\mathbb{K})$, for any commutative field $\mathbb{K}$, a generalized inverse of A is a matrix X such that $AXA=A$, $XAX=X$. Moore-Penrose is probably the most used generalized inverse. Other generalized inverses are the 1-matrices and the Drazin inverses.

Definition and properties of Drazin inverses are well known. The Drazin inverse of matrix A is usually denoted by A^D. In [2] and [3], differentiable families of Drazin inverses were studied.

The Drazin inverse of matrix A can be defined as the unique matrix which satisfies the following properties:

1. $A^DAA^D=A^D$.
2. $AA^D=A^DA$.
3. $A^{k+1}A^D=A^k$, $\forall k\geq i(A)$, where $i(A)$ is the nilpotency index of matrix A.

Recall that the nilpotency index of a matrix A is the least nonnegative integer k such that $\operatorname{rk}A^k=\operatorname{rk}A^{k+1}$.

If A is an invertible matrix, then $A^D=A^{-1}$. If not, assuming that A is a square matrix of order n and $rk\,A<n$, the Jordan reduced form of matrix A

is diag (J, N) where $\det(J) \neq 0$ and N is a nilpotent matrix. In this case, if $A = S\text{diag}\,(J, N)S^{-1}$, then $A^D = S^{-1}\text{diag}\,(J^{-1}, 0)S$, as can be easily checked.

In particular, similar matrices have similar Drazin inverses, but the converse is not true.

In Section above, in any of the strata there were matrices having the same Segre characteristic for the invertible part of their reduced Jordan form. Since Segre characteristics of an invertible matrix and that of its inverse coincide, we can conclude the following statement.

Proposition 2 *Two matrices $A, A' \in M_m(\mathbb{C})$ belong to the same stratum (with respect to the stratification obtained in the preceding Section) if, and only if, their Drazin inverses have the same Segre characteristics.*

5. Application to the space of matrices defining linear systems

As seen in Section 2, if we consider a linear system $\dot{x}(t) = Ax(t)$, the stratification above can be used, and all systems with matrix $A \in M_n(\mathbb{C})$ with the same Segre characteristics for the invertible part of the Jordan reduced form will belong to the same class. In the case where a singular linear system defined by a pair of matrices $(E, A) \in \mathcal{M} = M_n(\mathbb{C}) \times M_n(\mathbb{C})$ is considered, we can first obtain its standard reduced form (slow and fast subsystems) and then use the stratification above.

The pairs of matrices defining two singular dynamical systems $E\dot{x}(t) = Ax(t)$, $E'\dot{x}(t) = A'x(t)$ are in the same stratum if, and only if, matrices A and A' have invertible block in the Jordan canonical form of the same order and the Segre characteristics of these two matrices coincide.

6. Conclusions

A new stratification in the space of matrices defining singular linear systems is obtained. It is based in considering the matrices multiplying the state variables having the same Drazin inverse as belonging to a same stratum. This stratification can be generalized to the set of matrices defining the subsystems of a switched linear system (a system consisting of different subsystems of linear equations and a rule providing the switching between them), the strata being the Cartesian products of strata defined above (product stratification) and again a finite stratification.

References

[1] V.I. Arnold, On matrices depending on parameters. *Russian Math. Surveys* **26: 2**, pp. 29-43 (1971).

[2] S.L. Campbell, C.D. Meyer Jr. and N.J. Rose, Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients, *SIAM Journal on Applied Mathematics* **31** No. 3, pp. 411-425 (1976).

[3] J. Clotet and M.D. Magret, Familias diferenciables de inversas de Drazin. E-prints UPC (2005).

[4] L. Dai, Singular control systems, *Lecture Notes in Control and Information Sciences* **118** (1989).

[5] A. Edelman, E. Elmroth and B. Kagstrom, A geometric approach to perturbation theory of matrices and matrix pencils. Part II: A stratification-enhanced staircase algorithm, *SIAM J. Matrix Anal. Appl.* **20**, No. 3, pp. 667-699 (1999).

[6] M. I. García-Planas, Differentiability of family of generalized-inverses of a differentiable family of matrices. *Proceedings of the 8th WSEAS International Conference on Applied Mathematics*, pp. 165-169 (2006).

[7] M. I. García-Planas and M.D. Magret, Stratification of linear systems. Bifurcation diagrams for families of linear systems. *Linear Algebra and its Applications* **297**, pp. 23-56 (1999).

[8] C.G. Gibson, Regularity of the Segre Stratification. *Math. Proc. Camb. Phil. Soc.* **80**, pp. 91-97 (1976).

[9] S. Johansson, Stratification of matrix pencils in systems and control: theory and algorithms, Licentiate Thesis (2005).